Adiabatic Hadamard States for Dirac Quantum Fields on Curved Space

Stefan Hollands*
Department of Mathematics, University of York,
York YO10 5DD, UK

Abstract

In this paper we propose a definition of quasifree Hadamard states for spinor fields on a curved space-time by specifying the Polarisation Set of the two-point function. We prove that the thermal equilibrium state on an ultrastatic space-time is Hadamard. We then construct an adiabatic vacuum state on a general globally hyperbolic Lorentz manifold using a factorisation of the spinorial Klein-Gordon operator. This state is pure. In what constitutes the main part of the paper, we show that it is also Hadamard. As a side result, we obtain the propagation of singularities of the spinorial Klein-Gordon operator. Some notation and results are collected in the Appendix.

*Electronic mail: hollands@physik.hu-berlin.de

I Introduction

In many cases of physical interest, for example the early stages of the universe or stellar collapse, one faces the problem of constructing quantum field theories on a non-static curved space-time. As a preparation for more complicated models such as QED, we shall study adiabatic quantum states for a free Dirac field on a general globally hyperbolic space-time.

We find it convenient to work in the algebraic framework of quantum field theory, which started with the work of R. Haag and D. Kastler [1], for an overview see [2]. In this approach one deals with a net of C^*-algebras $\{\mathcal{A}(\mathcal{O})\}_{\mathcal{O}\subset M}$ of observables localised in a space-time region $\mathcal{O} \subset M$. The algebra $\mathcal{A} = \overline{\cup_{\mathcal{O}\subset M}\mathcal{A}(\mathcal{O})}$ is called the 'quasilocal algebra'. In this approach, quantum states are positive normalised linear functionals on $\mathcal{A}$. One of the major difficulties of QFT on curved space-times is to pick out physically reasonable states. This is because, due to the absence of space-time symmetries, there is no analogue of the spectrum condition, which is a powerful tool to single out physical states in Minkowski space.

Hadamard states are thought to be good candidates for physical states at least for free quantum field theories in curved space-time. They allow for a point-splitting renormalisation of the stress-energy tensor $T_{\mu\nu}$ [3]. R. Verch [4] has shown that in case of the quantised Klein-Gordon field, Hadamard states are quasi-equivalent and he has also shown local definiteness in the sense of Haag et. al. [5].

Numerous papers have been devoted to the study of Hadamard states for free scalar field theories, especially since the important discovery, of M. Radzikowski [6], that (quasifree) Hadamard states can be characterised by the Wave Front Set of their two-point function. This made it possible, among other things, to construct Wick products of free fields or to adapt the Epstein-Glaser approach to renormalisation to theories on a curved background, using the powerful tools microlocal analysis and the theory of pseudo-differential operators (PDO's) [7].

It seems that less work has been done for spinor fields in this direction. This is not due to conceptual problems but rather because the microlocal analysis of multicomponent fields is technically more involved. The extension of the techniques developed for scalar fields to multicomponent fields seems desirable. We propose to characterise Hadamard states in terms of their Polarisation Sets, a concept which is a refinement of the Wave Front Set of a distribution [8].

If the space-time in question has a timelike Killing vector field, one can fix a ground state by projecting on the positive frequent solutions of the Dirac equation. This strategy is however not appropriate on a general glob-

ally hyperbolic space-time, because positive and negative frequent modes (determined at an instant of time) will mix when propagated. Or, to put it differently, the Hamiltonian is not diagonal with respect to positive and negative frequent modes. Instead, they have to be determined dynamically off the Cauchy surface. This is achieved by a factorisation of the spinorial Klein-Gordon operator into positive and negative frequency parts, which has been considered before in W. Junker's work on adiabatic vacuum states for the Klein-Gordon field [9]. Making use of a characterisation of quasifree states on the CAR-algebra [10], we obtain a pure state which approximates the vacuum state of an ultrastatic space-time. For this reasons, we will call it an adiabatic ground state. The main result (Thm. VI.1) of our work is that it is of Hadamard type. As a warm-up exercise, we first obtain ground and thermal equilibrium states on ultrastatic spacetimes, and show that these are Hadamard. The analysis relies heavily on various results from microlocal analysis and the theory of PDO's, especially the propagation of singularities theorem. As side-result, the propagation of singularities for the spinorial Klein-Gordon operator is obtained in the proof of Thm. VI.1. Our techniques may be used to prove that the states introduced by L. Parker [11] for a free Dirac field on a Robertson-Walker space-time are of Hadamard type. This was already conjectured in [12] and will be discussed in more detail in a subsequent paper. Recently, adiabatic states describing thermal equilibrium have been invented for the free Klein-Gordon field on a Robertson-Walker space-time [13]. It would be interesting to ask whether this can be done for the Dirac field as well.

Our work is organised as follows. In Sec. II we define the local algebras of observables corresponding to a free Dirac field and characterise quasifree states. In Sec. III we give present the Hadamard condition for the Dirac field. In the next three sections, we give a definition of ground, KMS and adiabatic states and prove that the Hadamard condition holds in all three cases. We have shifted some results and definitions from microlocal analysis into the Appendix, where further reference can be found.

It should be noted that independent of our work, M. Radzikowski has investigated a similar definition of Hadamard states for spinor fields and also considered the propagation of singularities [14]. We are very grateful to him for making his results avaliable to us prior to publication.

II Local algebras and quasifree states

It was shown in [15] how to associate a net of algebras of observables to the free Dirac field on a globally hyperbolic spacetime. For convenience of the

reader we shall briefly sketch the main line of argument and recall how a certain class of states on this algebra, the so-called quasifree states, can be characterized. This will be used in order to construct quasifree Hadamard states in the later sections.

Let us start with some notation. Throughout this article (M, g) will denote a 4-dimensional globally hyperbolic Lorentz manifold of signature $(+,-,-,-)$. Globally hyperbolic manifolds can be foliated by 3-dimensional spacelike hypersurfaces Σ, implying that topologically $M = \Sigma \times \mathbf{R}$. It can be shown that every globally hyperbolic Lorentz manifold admits a spin-structure, i.e. a $2:1$ cover of the frame-bundle with a $Spin(3,1)$ principal fibre-bundle $\mathcal{P}$ [16]. Spinors resp. cospinors are sections in the associated vector bundles

$$DM = \mathcal{P} \times_\tau E, \quad D^*M = \mathcal{P} \times_{\tau^*} E^*,$$

where τ denotes the fundamental representation of $Spin(3,1)$ on the representation space E and τ^* is the conjugate representation on the dual E^*. Gamma matrices on a spin-manifold are defined to satisfy the usual anti-commutation relations[1] $\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu}$ and contraction of vector indices with gamma matrices is denoted by a slash. The pull-back of the Levi-Civita-connection to the spin-cover of (M, g) is called the spin-connection and will be denoted by ∇. The connection coefficients can be expressed in terms of the Christoffel symbols of the Levi-Civita connection. In a local frame we have

$$\nabla_\mu = \partial_\mu - \frac{1}{4}\gamma^a\gamma^b\Gamma_{ab\mu}.$$

We consider the Dirac equation for smooth spinor and cospinor fields u resp. v,

$$(i\gamma^\mu\nabla_\mu - m)u = 0, \quad v(-i\gamma^\mu\nabla_\mu - m) = 0 \tag{1}$$

and construct the algebra $\mathrm{CAR}(\mathcal{K}, \Gamma)$ associated to the space of classical solutions to this equation. Let $L^2(\Sigma, DM)$ and $L^2(\Sigma, D^*M)$ be the spaces of square integrable spinor resp. cospinor fields equipped with the scalar product

$$\langle u_1, u_2\rangle = \int_\Sigma (\bar{u}_1 \not{n} u_2)(x)|h(x)|^{1/2}d^3x,$$

[1] Indices from the Greek alphabet denote components in a local chart, while indices from the beginning of the Roman alphabet label components in a local frame. Letters from the middle of the Roman alphabet mean spacelike components with respect to a local chart of Σ.

where n is the forward directed normal vector field to Σ, bar denotes the Dirac conjugate and h is the induced metric on Σ. Let $\mathcal{K} = L^2(\Sigma, DM) \oplus L^2(\Sigma, D^*M)$ and define the antiunitary involution ($\Gamma^2 = 1$) on $\mathcal{K}$,

$$\Gamma = \begin{bmatrix} 0 & \iota^{-1} \\ \iota & 0 \end{bmatrix},$$

where $\iota : DM \to D^*M$ is the antilinear map of Dirac conjugation. $\mathcal{K}$ is the space of initial data to the Dirac equation for particle and antiparticle fields. The algebra CAR$(\mathcal{K}, \Gamma)$ is the unique C^*-algebra generated by elements $A(F),\ F \in \mathcal{K}$ obeying the anticommutation relations

$$\{A(F_1), A(F_2)\} = \langle \Gamma F_1, F_2 \rangle_{\mathcal{K}},$$

and with the property $A(F)^* = A(\Gamma F)$, see e.g. [10].

If U is an isometry on $\mathcal{K}$ commuting with Γ, then there exists a $*$ -automorphism α_U of the CAR-algebra satisfying

$$\alpha_U(A(F)) = A(UF).$$

α_U is called the Bogoliubov automorphism corresponding to U. In order to find the observables located in a space-time region one makes use of the one-to-one correspondence between initial data and solutions to the Dirac equation. More formally, $\mathcal{K}$ is isomorphic to the completion of the space of solutions $C_0^\infty(M, DM)/kerS \oplus C_0^\infty(M, D^*M)/kerS^*$, where S resp. S^* are the uniquely defined causal propagators for Eqs. (1), see [15] (from now on we drop the star on S). The isomorphism is given by restricting a solution the Cauchy surface, $Su \oplus Sv \to \rho Su \oplus \rho Sv$, ρ denoting the restriction operator to the surface Σ. The isomorphism shows in particular that the construction is independent of a particular choice for Σ. The field operators for particle/antiparticles are given by

$$\Psi(v) = A\left(\begin{bmatrix} 0 \\ i\rho Sv \end{bmatrix}\right), \quad \bar{\Psi}(u) = A\left(\begin{bmatrix} i\rho Su \\ 0 \end{bmatrix}\right),$$

with the usual anticommutator

$$\{\Psi(v), \bar{\Psi}(u)\} = iS(v, u). \tag{2}$$

The local algebras of observables are

$$\mathcal{A}(\mathcal{O}) = C^*\{A(F), \quad F \in \mathcal{K}, \quad supp(F) \subset \mathcal{O}\}^{\text{even}},$$

where we mean the even part of the C^*-algebra generated by these elements, i.e. the fixalgebra of the even/odd automorphism generated by $F \to -F$.

A state on $\mathcal{A}$ is called *gauge invariant*, if $\omega = \omega \circ \alpha_\theta$, where α_θ is generated by

$$F = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} \to \begin{bmatrix} e^{i\theta} f_1 \\ e^{-i\theta} f_2 \end{bmatrix}$$

Moreover, a state ω such that

$$\omega(A(F_1) \dots A(F_n) A(G_1) \dots A(G_n)) = det\left(\langle \Gamma F_i, T G_j \rangle_{\mathcal{K}}\right)_{i,j=1,\dots,n}$$

for some operator $0 \leq T \leq 1,\ T + \Gamma T \Gamma = 1$ is called *quasifree*. These states give rise to a Fock-representation π of the CAR via a the GNS-construction,

$$\pi(A(F)) = a(T^{1/2} F) + a(\Gamma(1-T)^{1/2} F)^*, \tag{3}$$

where the a's denote the creation and annihilation operators on the fermionic Fock-space $\mathcal{F}$ over $\mathcal{K}$. Conversely, any operator T with the above properties determines a quasifree state on $\mathcal{A}$ via Eq. (3). Such a state ω is pure if and only if T is a projection. Moreover, a quasifree state is gauge invariant if and only if $T = B \oplus \iota(1-B)\iota^{-1}$ for some operator $0 \leq B \leq 1$.

III The Hadamard condition for spinor fields

Hadamard states were introduced in order to define the expectation value of the stress energy tensor of a linear scalar quantum field on a curved space time. Their two-point functions have a singularity of a particular form on the diagonal, according to the dimensionality of spacetime. For a mathematically rigorous definition see Kay and Wald [17]. In this article we prefer to work with a definition that emphasizes the microlocal properties of Hadamard states. Both characterisations are known to be equivalent since the work of M. Radzikowski [6], at least for scalar fields. Due to the vector character of Dirac fields, the definition is however technically more involved than for scalar fields. For a brief explanation of the technical ingredients and a few important results we refer to the Appendix.

Let $u \in C_0^\infty(M, DM)$ and $v \in C_0^\infty(M, D^*M)$. We denote by

$$\Lambda^{(+)}(v,u) = \omega(\Psi(v)\bar{\Psi}(u)), \quad \Lambda^{(-)}(v,u) = \omega(\bar{\Psi}(u)\Psi(v)) \tag{4}$$

the spatio-temporal two-point functions of a state ω, which we assume to be distributions. In order to lighten the notation, let us introduce the set $C \subset T^*(M \times M)$ of all pairs $(x_1, \xi_1) \sim (x_2, \xi_2)$, where $\sim$ means that x_1 and x_2 can be joined by a null-geodesic c such that $\xi_1 = \dot{c}(0)$ and $\xi_2 = \dot{c}(1)$. For later use, we also introduce the sets $N_\pm \subset T^*M$ of all future/past directed covectors ξ, i.e. satisfying $g(\xi, \xi) \geq 0$ and $\pm \xi^0 \geq 0$.

Definition III.1. A quasifree state ω is said to be 'Hadamard' if the (primed) Polarisation sets $\mathrm{WF}'_{pol}(\Lambda^{(\pm)})$ of its spatio-temporal two-point functions are of the form

$$\big\{(x_1,\xi_1,x_2,\xi_2,w) : (x_1,\xi_1) \sim (x_2,\xi_2),\ \pm\xi_1^0 \geq 0,\ w = \lambda \not\xi_1 \mathcal{I}_\nabla(x_1,x_2)\big\}, \qquad (5)$$

where

$$w \in D_{x_1}M \otimes D^*_{x_2}M, \quad \xi_1 \in T^*_{x_1}M, \quad \xi_2 \in T^*_{x_2}M, \quad \lambda \in \mathbf{R}$$

and $\mathcal{I}_\nabla$ is the parallel transport function on the spin-bundle.

To prove that a given quasifree state is of Hadamard type we only have to investigate the Polarisation Set of its two-point function. This shall be done for ground states and KMS states on ultrastatic space-times and adiabatic vacuum states on general globally hyperbolic space-times. Note that the above definition especially implies the relation

$$\mathrm{WF}'(\Lambda^{(\pm)}) = C \cap (N_\pm \times N_\pm)$$

for the Wave Front Set.

IV Ground states

As we mentioned before, ground states can be defined properly on ultrastatic spacetimes only. For the free Dirac field, such a state can be constructed quite easily. An ultrastatic spacetime is by definition a spacetime $M = \mathbf{R} \times \Sigma$ with a line element $ds^2 = dt^2 - h_{ij}dx^i dx^j$, the spatial part of the metric being time independent. Eqs. (1) can be written as

$$i\partial_t F = \begin{bmatrix} H & 0 \\ 0 & -\iota H \iota^{-1} \end{bmatrix} F = \widetilde{H} F$$

on $\mathcal{K}$, where

$$H = -i\gamma^0\gamma^j\nabla_j + \gamma^0 m.$$

H is an essentially self-adjoint operator on $L^2(\Sigma, DM)$. Let us denote by $E_\pm$ the projections on the positive/negative spectral subspace of H. A projector T with the property $T + \Gamma T \Gamma = 1$ is the given by

$$T = \begin{bmatrix} E_+ & 0 \\ 0 & \iota E_- \iota^{-1} \end{bmatrix}.$$

By the remarks of the preceding section, it defines a pure state on $\mathcal{A}$, which is also gauge invariant. The definition of T takes into account that particle states move forward in time with positive energy whereas antiparticle states move backwards in time with positive energy or move forwards in time with negative energy. It can be seen that the above state is the Fock-state which has the lowest energy among all Fock-states where the energy operator can be defined and has positive energy. The state defined by T is therefore the uniquely defined ground state. The following theorem is a special case of Thm. VI.1.

Theorem IV.1. *The ground state on an ultrastatic space-time* (M, g) *is Hadamard in the sense of Def. III.1.*

V Thermal equilibrium states

On ultrastatic space-times, thermal equilibrium states can be defined. Such a state is a KMS state [18] whose modular automorphism group coincides with the dynamics of the system. We define

$$T_\beta = \frac{\exp(-\beta \widetilde{H}/2)}{2\cosh(\beta \widetilde{H}/2)} = \frac{\exp(-\beta H/2)}{2\cosh(\beta H/2)} \oplus \iota \frac{\exp(\beta H/2)}{2\cosh(\beta H/2)} \iota^{-1}.$$

One sees that $\Gamma T_\beta \Gamma + T_\beta = 1$ and $0 \leq T_\beta \leq 1$, so it defines a quasifree state ω_β and by the GNS-construction a representation $(\pi_\beta, \Omega_\beta, \mathcal{F}_\beta)$ of $\mathcal{A}$. The expressions for the modular Hamiltonian and the modular conjugation can be read of from

$$\pi_\beta(X)^* \Omega_\beta = J e^{-\beta \, d\mathcal{F}(\widetilde{H})/2} \pi_\beta(X) \Omega_\beta, \quad X \in \mathcal{A},$$

where

$$J(F_1 \wedge \cdots \wedge F_n) \stackrel{\text{def}}{=} \Gamma F_n \wedge \cdots \wedge \Gamma F_1,$$

and $d\mathcal{F}$ is the second quantisation functor. This confirms that we have indeed defined a thermal equilibrium state at inverse temperature β, since the modular automorphism coincides with the time-evolution of the system.

Theorem V.1. *The state induced by the operator* T_β *is Hadamard in the sense of Def. III.1.*

Proof. From the definition of the spation temporal two-point functions, Eq. (4) and T_β one calculates

$$\Lambda_\beta^{(\pm)}(v, u) = \langle \rho S \bar{v}, Q_\pm \rho S u \rangle, \tag{6}$$

where

$$Q_\pm = \frac{\exp(\pm\beta H/2)}{2\cosh(\beta H/2)}.$$

Since S is a solution to the Dirac equation in both entries, it is immediately clear that

$$Q_\pm \rho S = \rho q_\pm(i\partial_t)S \tag{7}$$

modulo $\mathcal{L}^{-\infty}$, where $q_\pm \in \mathcal{S}^0(\mathbf{R})$ are given by

$$q_\pm(\lambda) = \frac{e^{\pm\beta\lambda/2}}{2\cosh(\beta\lambda/2)}.$$

Let $\chi_\pm$ be smooth functions on the real line equal to the characteristic functions of $\mathbf{R}_\pm$ except for a small neighbourhood of the origin. Now $\chi_\mp q_\pm$ is a symbol in $\mathcal{S}^{-\infty}(\mathbf{R})$, hence $\chi_\mp(i\partial_t)q_\pm(i\partial_t) \in \mathcal{L}^{-\infty}(\mathbf{R})$ and we can conclude that (identifying the operators with the corresponding distribution kernels)

$$\mathrm{WF}'(\chi_\mp q_\pm(i\partial_t)) \subset \{(\vec{x}, t, \vec{\xi}, 0; \vec{x}, t, \vec{\xi}, 0) : \quad (\vec{x}, \vec{\xi}) \in T^*\Sigma,\ t \in \mathbf{R}\}. \tag{8}$$

By Thm. VII.3 and Eq. (8),

$$\begin{aligned} \mathrm{WF}'(\chi_\mp(i\partial_t)q_\pm(i\partial_t)S) &\subset \mathrm{WF}'(\chi_\mp q_\pm(i\partial_t)) \circ \mathrm{WF}'(S) \\ &\cup \mathrm{WF}_M(\chi_\mp q_\pm(i\partial_t)) \times M \\ &\cup M \times \mathrm{WF}_M(S) = \emptyset, \end{aligned}$$

implying $\chi_\mp(i\partial_t)q_\pm(i\partial_t)S \in C^\infty$. On the other side, the leading symbol of $\chi_\pm(i\partial_t)$ is equal to the characteristic function of the positive/negative real line. Hence by definition of the Wave Front Set and the pseudo-local property Eq. (19)

$$\mathrm{WF}'(q_\pm(\partial_t)S) \subset \mathrm{WF}'(S) \cap (\chi_\mp^{-1}(0) \times M) = C \cap (N_\pm \times N_\pm), \tag{9}$$

because $\mathrm{WF}'(S) = C$, which can be inferred from the corresponding property of the scalar causal propagator [6]. By Thm. VII.2 and Eqs. (7) and (9),

$$\mathrm{WF}'(Q_\pm \rho S) \subset (d\phi_1)^t(C \cap (N_\pm \times N_\pm)), \quad \mathrm{WF}'(S\rho') \subset (d\phi_2)^t(C), \tag{10}$$

where $\phi_1 : \Sigma \times M \to M \times M$ and $\phi_2 : M \times \Sigma \to M \times M$ are the embeddings. Taking into account Eq. (6), we see that by Thm. VII.3

$$\begin{aligned} \mathrm{WF}'(\Lambda_\beta^{(\pm)}) &= \mathrm{WF}'(S\rho' Q_\pm \rho S) \subset \mathrm{WF}'(S\rho') \circ \mathrm{WF}'(Q_\pm \rho S) \\ &\subset (d\phi_2)^t(C) \circ (d\phi_1)^t(C \cap (N_\pm \times N_\pm)) = C \cap (N_\pm \times N_\pm). \end{aligned}$$

To prove that equality holds in the above inclusion and that the polarisation has the prescribed form, we can proceed as in proof of Thm. VI.1. Hence we conclude that the state defined by T_β is indeed Hadamard, i.e. the positive and negative two-point functions have a Polarisation Set of the form Eq. (5). $\square$

VI Adiabatic vacuum states

It is known that ground states on general globally hyperbolic space-times cannot be constructed in the same way as in static ones, i.e. by just considering the spectrum of the Hamiltonian at an instant of time as above. This is related to the fact that there is no reasonable notion of positive and negative frequency modes in general globally hyperbolic space-times. Instead, they have to be extrapolated off the Cauchy surface. Following an idea by W. Junker [9], this might be achieved by considering a factorisation of the (spinorial) Klein-Gordon operator. The states obtained are in a sense an approximation of a ground state at some time t, taking the time-evolution of the metric in an infinitesimal neighbourhood of the Cauchy surface into account. They are called 'adiabatic'. The failure of the 'naive' ground state on a globally hyperbolic space-time to describe a physical state may be seen more formally from the fact that it can be shown not to satisfy the Hadamard condition. We will now construct such an adiabatic ground state for the Dirac field.

On each Cauchy surface $\Sigma(t)$, one can find an elliptic PDO $A(t)$ of order one satisfying

$$(-in^\mu\nabla_\mu + A(t)) \circ (in^\mu\nabla_\mu + A(t)) = g^{\mu\nu}\nabla_\mu\nabla_\nu + \frac{1}{4}R + m^2 \stackrel{\text{def}}{=} P \tag{11}$$

modulo $\mathcal{L}^{-\infty}$. A principal symbol of $A(t)$ is

$$a_1(\vec{x}, t, \vec{\xi}) = \sqrt{h(\vec{\xi}, \vec{\xi})}. \tag{12}$$

Such an operator may be constructed as an asymptotic expansion of its symbol [9]. We let A^{-1} be a parametrix (we drop the reference to t in places to lighten the notation). Then the operator

$$B \stackrel{\text{def}}{=} \mathrm{Re}(A^{-1}H)$$

is a selfadjoint elliptic PDO of order zero with principal symbol

$$b_0(\vec{x}, t, \vec{\xi}) = a_1(\vec{x}, t, \vec{\xi})^{-1} h_1(\vec{x}, t, \vec{\xi}) = \frac{\not{n}\not{\vec{\xi}}}{\sqrt{h(\vec{\xi}, \vec{\xi})}}.$$

By elementary continuity properties of PDO's, it extends to a bounded, self-adjoint operator on $L^2(\Sigma, DM)$ and we can employ the spectral calculus to define functions of this operator. Let $\chi_\pm$ be the characteristic functions of the positive resp. negative axis and

$$T_a \stackrel{\text{def}}{=} \chi^+(B) \oplus \iota\chi^-(B)\iota^{-1}.$$

Obviously, T_a is a projection and $\Gamma T_a \Gamma = 1 - T_a$. It defines a pure quasifree and gauge invariant state ω_a on $\mathcal{A}$. If (M, g) is an ultrastatic space-time, then $A = |H|$. In that case, T_a is the spectral projection corresponding to the positive part of the spectrum of H, i.e. our state is the ground state. In that sense our state should be understood as approximating the ground state. We now state the main theorem of this paper.

Theorem VI.1. *The state defined by T_a is Hadamard.*

Proof. The expressions for the positive/negative frequent two-point functions are

$$\Lambda_a^{(\pm)}(v, u) = \left\langle \rho S\bar{v}, \chi^\pm(\mathrm{Re}(A^{-1}H))\, \rho S u \right\rangle.$$

In the first part of the proof we will show that the Wave Front Set of the above two-point functions has the required form, whereas in the second part we will verify the claims about the polarisation specified in Def. III.1.

Let us set $f^\pm(\lambda) = (1 \pm \lambda)^{-2}\chi^\pm(\lambda)$. We first want to show that $f^\pm(B)$ are PDO's. To this end notice that

$$f^\pm(B)u = \frac{1}{2\pi i}\oint_{\mathcal{C}_\pm} \frac{1}{(1 \pm z)^2} R(z, B)u\, dz,$$

where $\mathcal{C}_\pm$ is a contour around the positive part of the spectrum, keeping away from the set $[\varepsilon, C]$ resp. $[-C, -\varepsilon]$ for some $C, \varepsilon > 0$, to be specified in a second. $R(z, B)$ is the resolvent of B. We can always choose such contours, because B is bounded and zero cannot be an accumulation point in the spectrum, the latter fact following easily from the existence of a parametrix to B. By Lemma VII.1 the resolvent is given by $r(z, \vec{x}, \vec{D})$, where r is a symbol in $\mathcal{S}^0$ depending smoothly on z in any open region of the complex plane whose closure does not intersect $\mathit{spec}\, B \cup [-C, -\varepsilon] \cup [\varepsilon, C]$, for some $C, \varepsilon > 0$. Because $(1 \pm z)^{-2}$ has no pole for $\mathrm{Re} z \geq 0$ resp. ≤ 0, it follows that $f^\pm(B)$ is the PDO corresponding to the symbol

$$\sigma(f^\pm(B)) = \frac{1}{2\pi i}\oint_{\mathcal{C}_\pm} \frac{1}{(1 \pm z)^2} r(z, \vec{x}, \vec{\xi})\, dz,$$

Let us set

$$K_1^{\pm} = iS\rho' \mathrm{Re}(1 \pm A^{-1}H)\rho S, \quad K_2^{\pm} = \not{n} f^{\pm}(\mathrm{Re}(A^{-1}H))\not{n}.$$

Using the identity $\rho S \rho' = -i\not{n}$ (see Dimock, [15]), it follows that we can write

$$\Lambda_a^{(\pm)}(v,u) = \left\langle \rho K_1^{\pm}\bar{v}, K_2^{\pm}\rho K_1^{\pm}u \right\rangle.$$

Suppose for the moment that the following inclusions hold:

$$\mathrm{WF}'(\Lambda_a^{(\pm)}) \subset \mathrm{WF}'(K_1^{\pm}\rho') \circ \mathrm{WF}'(K_2^{\pm}\rho K_1^{\pm}), \tag{13}$$
$$\mathrm{WF}'(K_1^{\pm}) \subset C \cap (N_{\pm} \times N_{\pm}). \tag{14}$$

As above, let $\phi_1 : \Sigma \times M \to M \times M$ and $\phi_2 : M \times \Sigma \to M \times M$ be the embeddings. It follows from the pseudo-local property Eq. (19) of the operator $K_2^{\pm}$ and the behaviour of Wave Front Sets under restriction (Thm. VII.2) that

$$\begin{aligned} \mathrm{WF}'(\Lambda_a^{(\pm)}) &\subset \mathrm{WF}'(K_1^{\pm}\rho') \circ \mathrm{WF}'(\rho K_1^{\pm}) \\ &\subset (d\phi_1)^t(C \cap (N_{\pm} \times N_{\pm})) \circ (d\phi_2)^t(C \cap (N_{\pm} \times N_{\pm})) \\ &= C \cap (N_{\pm} \times N_{\pm}). \end{aligned} \tag{15}$$

The first inclusion Eq. (13) follows essentially from Thm. VII.3, but there are some technicalities. For details we refer the reader to [9], where a similar statement is demonstrated. To prove the second inclusion Eq. (14), we notice that

$$K_1^{\pm} = S\rho' A^{-1}\rho(A \pm in^{\mu}\nabla_{\mu})S + S(A^* \pm in^{\mu}\nabla_{\mu})\rho' A^{*-1}\rho S.$$

It follows from the Lichnerowicz formula for the square of the Dirac operator that

$$(g^{\mu\nu}\nabla_{\mu}\nabla_{\nu} + \frac{1}{4}R + m^2)S = 0, \tag{16}$$

both for the spinor and cospinor propagator. Now by Eq. (11), the definition of the Wave-Front Set and the equality $\mathrm{WF}'(S) = C$ it holds that

$$\begin{aligned} \mathrm{WF}'((A \pm in^{\mu}\nabla_{\mu})S) &\subset C \cap \sigma_1(A \mp in^{\mu}\nabla_{\mu})^{-1}(0) \\ \mathrm{WF}'(S(A^* \pm in^{\mu}\nabla_{\mu})) &\subset C \cap \sigma_1(A^* \mp in^{\mu}\nabla_{\mu})^{-1}(0). \end{aligned}$$

The main point here is that by Eq. (12) and the structure of C the right hand sides of the above inclusions are in fact equal to $C \cap (N_{\pm} \times N_{\pm})$. Also,

$$\begin{aligned} \mathrm{WF}'(S\rho' A^{-1}) &= \mathrm{WF}'(S\rho') \subset (d\phi_2)^t(C), \\ \mathrm{WF}'(A^{*-1}\rho S) &= \mathrm{WF}'(\rho S) \subset (d\phi_1)^t(C), \end{aligned}$$

since A is elliptic and by the restriction property of the Wave Front Set. Using the above relations it follows

$$\begin{aligned}
\mathrm{WF}'(S\rho' A^{-1}\rho(A \pm in^\mu \nabla_\mu)S) &\subset \mathrm{WF}'(S\rho' A^{-1}) \circ \mathrm{WF}'(\rho(A \pm in^\mu \nabla_\mu)S) \\
&\subset (d\phi_2)^t(C) \circ (d\phi_1)^t(C \cap (N_\pm \times N_\pm)) \\
&= C \cap (N_\pm \times N_\pm)
\end{aligned}$$

and similarly for $\mathrm{WF}'(S(A^* \pm in^\mu \nabla_\mu)\rho' A^{*-1}\rho S)$. This proves the inclusion Eq. (14). Now from the CAR, Eq. (2) we infer that $\Lambda_a^{(+)} + \Lambda_a^{(-)} = iS$, hence $C \subset \mathrm{WF}'(\Lambda_a^{(+)}) \cup \mathrm{WF}'(\Lambda_a^{(-)})$. From this it follows that in fact equality must hold in Eq. (15), i.e.

$$\mathrm{WF}'(\Lambda_a^{(\pm)}) = C \cap (N_\pm \times N_\pm).$$

Looking at the definition of Hadamard states, Def. III.1, we see that only the polarisation remains to be verified.

We aim at using the propagation of singularities theorem, Thm. VII.4. Since $\Lambda_a^{(\pm)}$ satisfy the Dirac equation, by the Lichnerowicz formula Eq. (16), they also satisfy

$$(P \otimes 1)\Lambda_a^{(\pm)} = (1 \otimes P)\Lambda_a^{(\pm)} = 0.$$

P is clearly of real principal type with $p_0(x, \xi) = -g(\xi, \xi)$. We might set $\tilde{p}_0(x, \xi) = 1$, so the Dencker connection Eq. (20) corresponding to the operator P becomes

$$\mathcal{D}_P = -2\xi^\mu \frac{\partial}{\partial x^\mu} + 2\Gamma_{\mu\nu\sigma}\xi^\nu \xi^\sigma \frac{\partial}{\partial \xi_\mu} + \frac{1}{2}\gamma^a \gamma^b \Gamma_{ab\mu}\xi^\mu - \Gamma^\mu_{\nu\mu}\xi^\nu.$$

To see the geometric meaning of the Dencker connection, we let $\pi : T^*M \to M$ be the projection from the cotangent bundle to the base. We may then rewrite the Dencker connection as the pull-back to T^*M of the connection $\nabla + d\log|g|^{-1/4}$, the natural connection in the bundle $DM \otimes L^{1/2}$, taken in the direction of the Hamilton vectorfield,

$$\mathcal{D}_P = \pi^*(\nabla + d\log|g|^{-1/4})_{\mathcal{H}_q}.$$

Here $L^{1/2}$ is the line bundle of half-densities over M and

$$\mathcal{H}_q = -2\xi^\mu \frac{\partial}{\partial x^\mu} + 2\Gamma_{\mu\nu\sigma}\xi^\nu \xi^\sigma \frac{\partial}{\partial \xi_\mu}$$

is the Hamilton vectorfield corresponding to $q(x, \xi) = -g(\xi, \xi)$, generating null-geodesics in M. Sections over integral curves of $\mathcal{H}_q$, annihilated by $\mathcal{D}_P$

are thus pull-backs to T^*M of sections in DM over null-geodesics which are parallel with respect to ∇. Hence, two elements (x_1, ξ_1, u_1) and (x_2, ξ_2, u_2) of π^*DM are in the same Hamiltonian orbit if $(x_1, \xi_1) \sim (x_2, \xi_2)$ and if $u_1 = \lambda \mathcal{I}_\nabla(x_1, x_2) u_2$ where $\mathcal{I}_\nabla$ denotes parallel transport in the spin-bundle along a null-geodesic and $\lambda \in \mathbf{R}$. From this one immediately finds the Hamiltonian orbits corresponding to the operators $P \otimes 1$ and $1 \otimes P$. Let

$$(x_1, \xi_1, x_2, \xi_2, w) \in T^*_{x_1}M \times T^*_{x_2}M \times D^*_{x_1}M \otimes D_{x_2}M$$

be in the Polarisation Set $\mathrm{WF}'_{pol}(\Lambda^{(\pm)}_a)$. Then if $w \neq 0$, the pair (x_1, ξ_1, x_2, ξ_2) must be in $\mathrm{WF}'(\Lambda^{(\pm)}_a)$ which was shown to be equals $C \cap (N_\pm \times N_\pm)$. By the propagation of singularities theorem, Thm. VII.4, the Polarisation Set must be a union of Hamiltonian orbits. This implies that

$$\begin{aligned} &(x_1, \xi_1, x_2, \xi_2, w) \in \mathrm{WF}'_{pol}(\Lambda^{(\pm)}_a) \\ \Leftrightarrow\ &(x_1, \xi_1, x_1, \xi_1, w\mathcal{I}_\nabla(x_2, x_1)) \in \mathrm{WF}'_{pol}(\Lambda^{(\pm)}_a) \\ \Leftrightarrow\ &(x_2, \xi_2, x_2, \xi_2, \mathcal{I}_\nabla(x_2, x_1)w) \in \mathrm{WF}'_{pol}(\Lambda^{(\pm)}_a). \end{aligned}$$

Now since the Polarisation Set is an invariant object, $w\mathcal{I}_\nabla(x_2, x_1)$ must transform like a gamma matrix under a change of gauge, hence one concludes that it must be proportional to $\not{\xi}_1$ and $\mathcal{I}_\nabla(x_2, x_1)w$ is proportional to $\not{\xi}_2$. But this is just the condition on the polarisation in Def. III.1. $\square$

VII Appendix

In what follows we shall need various results and definitions from the theory of distributions and the theory of pseudo-differential operators (PDO's). If not indicated otherwise, these may be found in standard textbooks, for example see [19, 20]. PDO's generalise classical differential operators in the sense that they give meaning to fractional powers of derivatives. They are defined in terms of so-called symbols. We shall not give the most general definition of a symbol here, since only a certain class of symbols is relevant to this work.

Definition VII.1. Let $\mathcal{O}$ be a subset of $\mathbf{R}^n$ and m be a real number. Then one defines the symbols of order m to be the set $\mathcal{S}^m(\mathcal{O}, \mathbf{R}^n)$ of all functions $a \in C^\infty(\mathcal{O}, \mathbf{R}^n)$ such that for every compact subset K of $\mathcal{O}$ the following estimate holds

$$\left| D^\alpha_x D^\beta_\xi a(x, \xi) \right| \leq C_{\alpha,\beta,K}(1 + |\xi|)^{m-|\beta|} \tag{17}$$

for all multi-indices α, β. D^α is $i^{|\alpha|}\partial_1^{\alpha_1} \ldots \partial_n^{\alpha_n}$. One also writes $\mathcal{S}^{-\infty} = \bigcap_m \mathcal{S}^m$.

There is the notion of the asymptotic expansion of a symbol which is an important tool for constructing PDO's. Suppose $a_j \in \mathcal{S}^{m_j}(\mathcal{O}, \mathbf{R}^n)$ for $j = 1, 2, \ldots$ with m_j monotonically decreasing to minus infinity. Then there exists $a \in \mathcal{S}^{m_0}(\mathcal{O}, \mathbf{R}^n)$ such that for all N

$$a - \sum_{j=0}^{N} a_j \in \mathcal{S}^{m_N}(\mathcal{O}, \mathbf{R}^n)$$

and a is defined modulo $\mathcal{S}^{-\infty}$. If $a \in \mathcal{S}^m(\mathcal{O}, \mathbf{R}^n)$ then the operator

$$Au(x) = \int e^{ix\xi} a(x, \xi) \hat{u}(\xi) \frac{d^n\xi}{(2\pi)^n}$$

is said to belong to $\mathcal{L}^m(\mathcal{O})$, the PDO's of order m. For the operator A one also writes $a(x, D)$. A is a continous linear operator from $\mathcal{D}(\mathcal{O})$ to $C^\infty(\mathbf{R}^n)$. By the Schwartz kernel theorem it is thus given by a distribution kernel $K_A \in \mathcal{D}'(\mathcal{O} \times \mathcal{O})$. K_A is smooth off the diagonal in $\mathcal{O} \times \mathcal{O}$ and smooth everywhere in $\mathcal{O} \times \mathcal{O}$ if $A \in \mathcal{L}^{-\infty}(\mathcal{O})$. Hence the asymptotic expansion of a symbol uniquely determines a PDO modulo smoothing operators. The above statement carries over to matrix valued symbols and PDO's without major changes. If a, b are symbols (possibly matrix valued), then the convolution product is defined to be

$$a \circ b(x, \xi) \sim \sum_{\alpha \geq 0} \partial_x^\alpha a(x, \xi) D_\xi^\alpha b(x, \xi) / \alpha! \tag{18}$$

A PDO A is said to be properly supported if the support of its kernel distribution has compact intersection with any set of the form $\mathcal{O} \times K$, $K \times \mathcal{O}$, K compact. If A, B are properly supported PDO's, then AB is a PDO with symbol $\sigma(AB) \sim \sigma(A) \circ \sigma(B)$. The principal symbol $\sigma_m(A)$ of a pseudo-differential operator of order m is the representer of its symbol in $\mathcal{S}^m(\mathcal{O}, \mathbf{R}^n)/\mathcal{S}^{m-1}(\mathcal{O}, \mathbf{R}^n)$. It transforms under a change of coordinates in such a way as to give a well-defined function on the cotangent bundle. By the composition law for symbols, it behaves multiplicatively under multiplication of two PDO's. If we have a smooth manifold M instead of $\mathbf{R}^n$ (or more generally a vector bundle $\mathbf{E}$), PDO's are defined to be the continuous operators on $\mathcal{D}(M, \mathbf{E})$ which have the above properties in each coordinte patch. A PDO A on a vector-bundle $\mathbf{E}$ is called elliptic of order m, if its symbol a is an invertible matrix for for $|\xi|$ greater than some constant and the estimate

$$\|a(x, \xi)^{-1}\| \leq C|\xi|^{-m}$$

holds. Note that ellipticity follows from the corresponding property of any principal symbol alone. Elliptic PDO's are invertible in the following sense:

Theorem VII.1. *If $A \in \mathcal{L}^m$ is elliptic, then there exists a parametrix $B \in \mathcal{L}^{-m}$ for A in the sense that $AB = BA = 1 \bmod \mathcal{L}^{-\infty}$. Such a parametrix is uniquely defined up to $\mathcal{L}^{-\infty}$.*

We come to the definition of the Polarization Set of a vector-valued distribution $u = (u^1, \cdots, u^k) \in \mathcal{D}'(\mathcal{O})^k$, $\mathcal{O}$ an open subset of $\mathbf{R}^n$. For details of the definition and the subsequent results see the paper by N. Dencker, [8].

Definition VII.2. The Polarisation Set of a vector-valued distribution u is defined as

$$\mathrm{WF}_{pol}(u) = \bigcap_{A \in \mathcal{L}^0,\ Au \in C^\infty} \mathcal{N}_A,$$

where $\mathcal{N}_A$ is the set of all $(x, \xi, w) \in T^*\mathcal{O} \times \mathbf{C}^k$ such that $\sigma_0(A)(x, \xi)w = 0$.

PDO's are pseudo-local in the sense that they do not enlarge the Polarisation Set of a distribution,

$$\mathrm{WF}_{pol}(Au) \subset \mathrm{WF}_{pol}(u). \tag{19}$$

From the transformation properties of the principal symbol it is clear that the definition can be carried over to the case of distributions with values in a vector-bundle $\mathbf{E}$. $\mathrm{WF}_{pol}(u)$ is then seen to be a linear subset of $\pi^*\mathbf{E}$, $\pi : T^*M \to M$ being the canonical projection in the fibres of the cotangent bundle. The ordinary Wave Front Set $\mathrm{WF}(u)$ of a distribution is obtained by taking all points $(x, \xi) \in T^*M$ such that the fibre over this point in $\mathrm{WF}_{pol}(u)$ is nontrivial. We quote two results on the behaviour Wave-Front Set under composition and restriction important to this work:

Theorem VII.2. *(Theorem 8.2.4 of [20]): Let $u \in \mathcal{D}'(M)$ and let $\phi : \Sigma \to M$ be a regularly embedded hypersurface in M. Then u can be restricted to Σ if $\mathrm{WF}(u) \cap N\Sigma = \emptyset$, $N\Sigma \subset T^*M$ being the conormal bundle to Σ. In this case $\mathrm{WF}(\rho u) \subset (d\phi)^t(\mathrm{WF}(u))$, where ρ denotes restriction and $(d\phi)^t$ is the transpose of the tangent map.*

Theorem VII.3. *(Theorem 8.2.14 of [20]). Let $A : \mathcal{D}(M_1) \to \mathcal{D}'(M_2)$ and $B : \mathcal{D}(M_2) \to \mathcal{D}'(M_3)$ linear continous maps. By the Schwartz Kernel theorem these correspond to distribution kernels $K_A \in \mathcal{D}'(M_2 \times M_1)$ and $K_B \in \mathcal{D}'(M_3 \times M_2)$. If*

$$\mathrm{WF}'(K_B)_{M_2} \cap \mathrm{WF}'(K_A)_{M_2} = \emptyset$$

then the convolution $B \circ A$ is well defined and

$$\begin{aligned} \mathrm{WF}'(K_{B\circ A}) &\subset (WF'(K_B) \circ \mathrm{WF}'(K_A)) \\ &\cup (M_1 \times \mathrm{WF}(K_A)_{M_3}) \cup (\mathrm{WF}(K_B)_{M_1} \times M_3) \end{aligned}$$

Here the prime means that one has to reverse the sign of the corresponding cotangent vector in the second slot, and for $K \in \mathcal{D}'(M_1 \times M_2)$,

$$\mathrm{WF}(K)_{M_2} = \{(x_2, \xi_2) : \quad (x_1, 0; x_2, \xi_2) \in \mathrm{WF}(K)\}.$$

There is an important theorem on the Polarisation Set of distributions satisfying $Au \in C^\infty$ for differential operators A of real principal type, which goes under the name 'propagation of singularities' [8]. Such operators are defined as follows:

Definition VII.3. A $k \times k$ system P of differential operators on a manifold M with principal symbol $p_0(x, \xi)$ is said to be of real principal type at (y, η) if there exists a $k \times k$ symbol $\tilde{p}_0(x, \xi)$ such that

$$\tilde{p}_0(x, \xi) p_0(x, \xi) = q(x, \xi) 1_k$$

in a neighbourhood of (y, η), where $q(x, \xi)$ is scalar and real.

One sets

$$\mathcal{Q}_P = \{(x, \xi) \in T^*M - (0) : \quad det(p_0(x, \xi)) = 0\}.$$

If f is a C^∞ function on $\mathcal{Q}_P$ with values in $\mathbf{C}^k$, then one defines

$$\mathcal{D}_P f = \mathcal{H}_q f + \frac{1}{2}\{\tilde{p}_0, p_0\} f + i \tilde{p}_0 p^s f, \tag{20}$$

$\mathcal{H}_q$ being the Hamiltonian vectorfield of q,

$$\begin{aligned} \mathcal{H}_q &= \partial_x q \, \partial_\xi - \partial_\xi q \, \partial_x, \quad \{\tilde{p}_0, p_0\} = \partial_\xi \tilde{p}_0 \, \partial_x p_0 - \partial_x \tilde{p}_0 \, \partial_\xi p_0, \\ p^s &= p_1 + \frac{1}{2} \partial_\xi D_x p_0, \quad \sigma(P) \sim p_0 + p_1 + p_2 + \ldots . \end{aligned}$$

One can prove that $\mathcal{D}_P$ is a partial connection along the Hamiltonian vectorfield restricted to $\mathcal{Q}_P$. Since there is some arbitrariness in the choice of the symbol $\tilde{p}$, the partial connection is not uniquely defined. One can however prove that the remaining arbitrariness is irrelevant in what follows.

Definition VII.4. A Hamilton orbit of a system P of real principal type is a line bundle $\mathcal{L}_P \subset \mathcal{N}_P | c$, where c is an integral curve of the Hamiltonian field on $\mathcal{Q}_P$ and $\mathcal{L}_P$ is spanned by the sections f satisfying $\mathcal{D}_P f = 0$, i.e. $\mathcal{L}_P$ is parallel with respect to the partial connection.

The following theorem goes under the name 'propagation of singularities' [8, 21].

Theorem VII.4. *Let P be as above and u a vector-valued distribution. Suppose $(x, \xi) \notin \mathrm{WF}(Pu)$. Then, over a neighbourhood of (x, ξ) in $\mathcal{Q}_P$, $\mathrm{WF}_{pol}(u)$ is a union of Hamilton orbits of P.*

In this paper, we use the resolvent $R(z, Q) = (Q - z)^{-1}$ of a an elliptic selfadjoint PDO of order zero (always assuming that z is not in $spec\, Q$). Let $U_{\varepsilon,C}$ be an open region in $\mathbf{C}$ whose closure does not intersect the set $[-C, -\varepsilon] \cup [\varepsilon, C] \cup spec\, Q$. The following lemma is needed in this work.

Lemma VII.1. *Let $Q = q(x, D) \in \mathcal{L}^0(M, \mathbf{E})$ be elliptic and self-adjoint, where $\mathbf{E}$ is a vector bundle over M. Then $R(z, Q) = r(z, x, D)$ for some symbol $r \in \mathcal{S}^0$, depending smoothly $z \in U_{\varepsilon,C}$ for some $\varepsilon, C > 0$.*

Proof. For simplicity, we shall only treat the case when $\mathbf{E} = \mathcal{O} \times \mathbf{C}^k$ and $\mathcal{O}$ an open subset in $\mathbf{R}^n$. The proof can be adapted to the general case by using a partition of unity on M and working in local trivialisations. Suppose in the following that $z \in U_{\varepsilon,C}$ is as described above. By choosing a suitable principal symbol, we can assume that $\|q_0(x, \xi)^{-1}\|$ is bounded for all ξ because Q is elliptic. Then $(q_0 - z)^{-1}$ will be in $\mathcal{S}^0(\mathcal{O} \times U_{\varepsilon,C}, \mathbf{R}^n)$ (we treat z like the variables x) for some $\varepsilon, C > 0$. If

$$(q - z)(x, D)(q_0 - z)^{-1}(x, D) = 1 - a_z(x, D),$$

then

$$\|\partial_z^l \partial_x^\alpha \partial_\xi^\beta a_z(x, \xi)\| \leq C_{l,\alpha,\beta}(1 + |\xi|)^{-1-|\beta|}$$

Let $e_z(x, \xi)$ be an asymtpotic sum of the symbols of

$$(q_0 - z)^{-1}(x, D)(a_z(x, D))^N, \quad N = 0, 1, \ldots.$$

depending thus smoothly on $z \in U_{\varepsilon,C}$. Then one has

$$(q - z)(x, D)e_z(x, D) = 1 - w_z(x, D),$$

and

$$\|\partial_z^l \partial_x^\alpha \partial_\xi^\beta w_z(x, \xi)\| \leq C_{l,\alpha,\beta}(1 + |\xi|)^{-N-|\beta|} \tag{21}$$

for any N. Multiplication with $R(z, Q)$ gives

$$R(z, Q) = e_z(x, D) + R(z, Q)w_z(x, D). \tag{22}$$

Now by the estimate Eq. (21), w_z is a continuous map from $H^s(\mathcal{O}, \mathbf{C}^k)$ to $H^t(\mathcal{O}, \mathbf{C}^k)$, uniformly in z and for any t, s, where we mean the Sobolev spaces. On the other hand, $R(z, Q)$ is a continuous operator in $H^t(\mathcal{O}, \mathbf{C}^k)$ uniformly in z, at least for $t = 0, 1, \ldots$. This is obvious for $t = 0$. For $t > 0$ one has, choosing an arbitrary positive elliptic operator L of order t,

$$\begin{aligned} \|R(z,Q)u\|_t &\leq C\|LR(z,Q)u\|_0 \leq C\|(Q-z)LR(z,Q)u\|_0 \\ &\leq C(\|u\|_t + \|[L,Q]R(z,Q)u\|_0) \leq C(\|u\|_t + \|R(z,Q)u\|_{t-1}), \end{aligned}$$

since $[L, Q]$ is a PDO of order $t-1$. The result follows by induction on t. Hence, for any t, s, the operator $R(z, Q)w_z(x, D)$ is continuous from $H^s(\mathcal{O}, \mathbf{C}^k)$ to $H^t(\mathcal{O}, \mathbf{C}^k)$, uniformly in z. It follows from well-known embedding theorems that it must hence be an operator defined by a kernel in $C^\infty(\mathcal{O} \times \mathcal{O}, \mathbf{C}^k \times \mathbf{C}^k)$, i.e. an operator in $\mathcal{L}^{-\infty}$. The same can be shown to hold for any z-derivative of this operator using the above estimates. Hence $R(z, Q)$ is a PDO with symbol $r \in \mathcal{S}^0(\mathcal{O} \times U_{\varepsilon,C}, \mathbf{R}^n)$, defined by Eq. (22). In particular, the symbol depends smoothly on $z \in U_{\varepsilon,C}$. $\square$

References

[1] Haag, R., Kastler, D.: An Algebraic Approach to Quantum Field Theory. J. Math. Phys. **5**, 848-861 (1964)

[2] Haag, R.: Local quantum physics: Fields, particles, algebras. Springer-Verlag, Berlin 1992

[3] Wald, R. M.: Quantum Field Theory in Curved Spacetime and Black Hole Thermodynamics. University of Chicago Press, Chicago 1994

[4] Verch, R.: Local definiteness, primarity and quasiequivalence of quasifree Hadamard quantum states in curved spacetime. Commun. Math. Phys. **160**, 507-536 (1994)

[5] Haag R., Narnhofer, H., Stein, U.: On Quantum Field Theory in Gravitational Background. Commun. Math. Phys. **94**, 219-238 (1984)

[6] Radzikowski, M.J.: Micro-Local Approach to the Hadamard Condition in Quantum Field Theory on Curved Space-Time. Commun. Math. Phys. **179**, 529-553 (1996)

[7] Brunetti, R., Fredenhagen, K.: Interacting quantum fields in curved space: Renormalisability of ϕ^4. gr-qc/9701048, Brunetti, R., Fredenhagen, K., Köhler, M.: The Microlocal Spectrum Condition and the

Wick Polynomials for Free Fields. Commun. Math. Phys. **180**, 633-652 (1996)

[8] Dencker, N.:On the propagation of Polarization Sets for Systems of Real Principal Type. J. Func. A. **46**, 351-372 (1982)

[9] Junker, W.: Adiabatic Vacua and Hadamard States for Scalar Quantum Fields on Curved Spacetime. Doctoral thesis, Hamburg 1995 hep-th/9507097

[10] Araki, H., Wyss, W.: Representations of the Canonical Anticommutation Relations. Helv. Phys. Acta **37**, 136-159 (1964)

[11] Parker, L.: Quantized Fields and Particle Creation in Expanding Universes II. Phys. Rev. **D3**, 346-356 (1971)

[12] Wellmann, M.: Adiabatic Vacuum States of the Dirac-Field on a Curved Space-Time, hep-th/9802087

[13] Trucks, M.: A KMS-Like State of Hadamard Type on Robertson-Walker Space-Times and its Time Evolution. Commun. Math. Phys. **197**, 387-404 (1998)

[14] Radzikowski, M.J.: Unpublished notes

[15] Dimock, J.: Dirac quantum fields on a manifold. Trans. Am. Math. Soc. **269**, 133-147 (1982)

[16] Geroch, R.: Spinor structures of space-times in General Relativity, II. J. Math. Phys. **11**, 343-348 (1970)

[17] Kay, B.S., Wald, R.M.: Theorems on the uniqueness and thermal properties of stationary, quasifree states on spacetimes with a bifurcate Killing Horizon. Phys. Rep. **207**, (1991)

[18] Haag R., Hugenholtz, N.M., Winnink, M.: 'On the Equilibrium States in Quantum Statistical Mechanics. Commun. Math. Phys. **94**, 219-238 (1984)

[19] Taylor, M.E.: Pseudodifferential Operators. Princeton University Press, Princeton 1981

[20] Hörmander, L.: The Analysis of Linear Partial Differential Operators. Springer-Verlag, Berlin 1983

[21] Duistermaat, J.J., Hörmander, L.: Fourier Integral Operators. Acta Mathematica **128**, 183-269 (1972)